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Global Optimization of Nonlinear Sums of Ratios

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The nonlinear sum of ratios problem (P) has several important applications. However, it is also a difficult problem to solve, since it generally possesses many local optima that are not global optima. In this article we present and show the convergence of an algorithm for finding a global optimal solution to problem (P). The algorithm uses a branch and bound search procedure that globally solves problem (P) by concentrating primarily on solving an equivalent outcome space version of the problem. The algorithm can be implemented by using standard convex programming methods.

1. INTRODUCTION

Consider the problem

$$vm = \min \sum_{i=1}^{p} \frac{n_i(x)}{d_i(x)}, \qquad \text{s.t. } x \in X, \tag{P}$$

where $p \geq 2$, n_i: $\Re^n \to \Re$ is a finite, convex function for each $i = 1, 2, \ldots, p$, d_i: $\Re^n \to \Re$ is a finite, concave function for each $i = 1, 2, \ldots, p$, X is a nonempty, compact convex set in $\Re^n$, and, for each $i = 1, 2, \ldots, p$, $n_i(x) \geq 0$ and $d_i(x) > 0$ for all $x \in X$. For each $x \in X$, let $f(x) = \sum_{i=1}^{p} n_i(x)/d_i(x)$. Notice that under the assumptions given, the global minimum vm of problem (P) is attained by at least one point in X. We refer to problem (P) as the *nonlinear sum of ratios* problem. This problem and special cases of this problem have attracted the interest of practitioners and researchers since the 1970s. This is because, from a practical point of

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view, problem (P) and its special cases have several important applications. Included among these applications, for example, are multistage stochastic shipping problems [1], government contracting problems [8], and bond portfolio optimization problems [15, 17]. From a research point of view, problem (P) poses important computational challenges. This is mainly due to the fact that, in general, problem (P) possesses many local optima that are not global optima; i.e., it is a global optimization problem. This is even true when $p = 2, X$ is a polyhedron, $n_1(x), n_2(x)$, and $d_2(x)$ are affine functions, and $d_1(x)$ is a constant [24].

Various global optimization algorithms have been proposed for solving problem (P) and special cases of problem (P). Many of these algorithms are for the *linear sum of ratios* problem, i.e., the problem obtained when X is a polyhedron and, for each $i = 1, 2, \ldots, p, n_i(x)$ and $d_i(x)$ are affine functions. For instance, Konno et al. [18] designed a finite, parametric simplex method-based algorithm that is appropriate for solving problem (P) when $p = 2, X$ is a polyhedron, and n_i and d_i are affine functions for each $i = 1, 2$. Later, Falk and Palocsay [10] developed an image space algorithm suitable for globally solving the same problem when $p \geq 2$. Since then, several algorithms using outer approximation, branch and bound, or parametric linear programming have been developed for solving various sums of ratios problems, many of which are special cases of problem (P) [7, 13, 14, 16, 20, 21].

To date, we are aware of two published algorithms that are specifically designed to globally solve the nonlinear sum of ratios problem (P). One of these algorithms, developed by Dur et al. [9], uses a branch and bound approach that can be implemented especially well for the linear sum of ratios problem. The other algorithm, by Freund and Jarre [11], uses an interior point method to underestimate the optimal value function of a convex program associated with problem (P).

For further reviews of sum of ratios problems, the reader is referred to Schaible [22, 23].

In this article, we present and show the convergence of an algorithm for finding a global optimal solution to problem (P). This algorithm uses a branch and bound search procedure. To economize the necessary computations, the algorithm globally solves problem (P) by concentrating primarily upon solving an equivalent outcome space problem that we call problem (Q). The algorithm can be implemented by using standard convex programming methods.

In Section 2, we show how to convert problem (P) into the equivalent outcome space problem (Q). The global optimization algorithm for problem (P) is presented in Section 3. In Section 4, convergence properties of the algorithm are shown, and techniques for initializing the algorithm are suggested. In the last section, some concluding remarks are given.

2. THE OUTCOME SPACE PROBLEM

In this section, we show how to convert problem (P) to a problem (Q) in the outcome space $\Re^{2p}$ of problem (P). The branch and bound algorithm to be developed will concentrate primarily on solving problem (Q) in order to globally solve problem (P). Since $2p$ is typically less than n, it is expected that the algorithm will benefit computationally from this approach.

Let $I = \{1, 2, \ldots, p\}$. For each $i \in I$, let N_i and M_i satisfy

$$N_i > \max_{x \in X} \ n_i(x) \tag{1}$$

and

$$M_i = \max_{x \in X} \ d_i(x), \tag{2}$$

respectively, where $N_i < +\infty$. Define the $2p$-dimensional rectangle H° by

$$H^\circ = \{(y, z) \in \Re^{2p} \mid 0 \leq y_i \leq N_i, 0 \leq z_i \leq M_i, i \in I\}, \tag{3}$$

and let $W \subseteq \Re^{2p}$ be given by

$$W = \{(y, z) \in H^\circ \mid y \geq n(x) \text{ and } z \leq d(x) \text{ for some } x \in X\},$$

where, for each $x \in X$, $[n(x)]^T = [n_1(x), n_2(x), \ldots, n_p(x)]$ and $[d(x)]^T = [d_1(x), d_2(x), \ldots, d_p(x)]$. It is easy to show that W is a nonempty, compact convex set in the outcome space $\Re^{2p}$ of problem (P). Notice also that the interior of W, denoted int W, is nonempty.

For each $(y, z) \in W$, let g: $\Re^{2p} \to \Re$ be given by

$$g(y, z) = \frac{\left[\sum_{i=1}^{p} y_i \left(\prod_{\substack{j=1 \\ j \neq i}}^{p} z_j\right)\right]^{1/p}}{\left(\prod_{i=1}^{p} z_i\right)^{1/p}},$$

and consider the outcome space problem

$$\min \ g(y, z), \qquad \text{s.t } (y, z) \in W. \tag{Q}$$

The following results give some properties of problem (Q). Let ∂W denote the boundary of W.

THEOREM 1. *Any global optimal solution for problem* (Q) *belongs to* ∂W.

Proof. Assume that $(y, z) \in W$. Suppose that $(y, z) \notin \partial W$. Then $(y, z) \in (\text{int } W)$. Therefore, we may choose a point $\bar{x} \in X$ such that $y > n(\bar{x})$ and $z < d(\bar{x})$, where, for each $i \in I$, $0 < y_i < N_i$ and $0 < z_i < M_i$. Let $\bar{y} = n(\bar{x})$ and $\bar{z} = d(\bar{x})$. Then, by the definition of W and the assumptions on $n_i(x)$

and $d_i(x)$, $i \in I$, $(\bar{y}, \bar{z}) \in W$. By the definition of g, since $0 \le n(\bar{x}) = \bar{y} < y$ and $0 < z < d(\bar{x}) = \bar{z}$, it follows that

$$g(\bar{y}, \bar{z}) = \left(\sum_{i=1}^{p} \frac{\bar{y}_i}{\bar{z}_i} \right)^{1/p} < \left(\sum_{i=1}^{p} \frac{y_i}{z_i} \right)^{1/p} = g(y, z).$$

Since $(\bar{y}, \bar{z}) \in W$, this implies that (y, z) cannot be an optimal solution for problem (Q). By the contrapositive, the desired result follows.

From Theorem 1, any global optimal solution for problem (Q) must belong to the boundary of W. Although, as we shall see below, problem (P) is equivalent to problem (Q), problem (P) may have no global optimal solutions on the relative boundary of X. For instance, let $p = n = 2$, let $X = \{(x_1, x_2) \mid 1 \le x_j \le 3, j = 1, 2\}$, and, for each $i = 1, 2$, let

$$n_i(x_1, x_2) = (x_i - 2)^2,$$

and

$$d_i(x_1, x_2) = x_{3-i}$$

in problem (P). Then the unique global optimal solution to problem (P) is given by $x^{*^T} = (2, 2)$ which does not lie on the boundary of X.

THEOREM 2. *Problem* (P) *is equivalent to problem* (Q) *in the following sense*: *If* (y^*, z^*) *is a global optimal solution for problem* (Q), *then any* $x^* \in X$ *such that* $n(x^*) \le y^*$ *and* $d(x^*) \ge z^*$ *is a global optimal solution for problem* (P), *and* $vm = [g(y^*, z^*)]^p = f(x^*)$. *Conversely, if* x^* *is a global optimal solution for problem* (P), *then* $(y^*, z^*) = [n(x^*), d(x^*)]$ *is a global optimal solution for problem* (Q), *and* $vm = [g(y^*, z^*)]^p = f(x^*)$.

Proof. Let (y^*, z^*) be a global optimal solution for problem (Q). Assume that $n(x^*) \le y^*$ and $d(x^*) \ge z^*$ for some $x^* \in X$. Then it is easy to see by the global optimality of (y^*, z^*) in problem (Q) that $n(x^*) = y^*$ and $d(x^*) = z^*$. Therefore, by the definition of g, $[g(y^*, z^*)]^p = f(x^*)$. Since $x^* \in X$, this implies that $g[(y^*, z^*)]^p = f(x^*) \ge vm$. Suppose that $f(x^*) > vm$. By definition of vm, this implies that $f(x^*) > f(x)$ for some $x \in X$. Let $y = n(x)$ and $z = d(x)$. Then $(y, z) \in W$, and $f(x) = [g(y, z)]^p$. Summarizing, we have found a point $x \in X$ and a point $(y, z) \in W$ such that

$$[g(y, z)]^p = f(x) < f(x^*) = g[(y^*, z^*)]^p.$$

Since this implies that (y^*, z^*) is not a global optimal solution for problem (Q), the supposition that $f(x^*) > vm$ must be false. Therefore, $f(x^*) = vm$. Since $x^* \in X$ and $g[(y^*, z^*)]^p = f(x^*)$, this implies that x^* is a global optimal solution for problem (P) and $vm = [g(y^*, z^*)]^p = f(x^*)$.

Let x^* be a global optimal solution for problem (P), and let $(y^*, z^*) = [n(x^*), d(x^*)]$. Then it is easy to see that $(y^*, z^*) \in W$ and that $[g(y^*, z^*)]^p = f(x^*) = vm$. Suppose that (y^*, z^*) is not a global optimal solution for problem (Q). Then, for some $(y, z) \in W, [g(y^*, z^*)]^p > [g(y, z)]^p$. Since $(y, z) \in W$, we may select a vector $x \in X$ such that $y \geq n(x)$ and $z \leq d(x)$. Since $n(x) \geq 0$ and $d(x) > 0$, this implies that $f(x) \leq g[(y, z)]^p$. Summarizing, we have found a point $x \in X$ and a point $(y, z) \in W$ such that

$$f(x) \leq g[(y, z)]^p < g[(y^*, z^*)]^p = f(x^*).$$

Since this implies that x^* is not a global optimal solution for problem (P), the supposition that (y^*, z^*) is not a global optimal solution for problem (Q) is false, and the proof is complete.

In [6], Cambini et al. give a result similar to Theorem 2. Their result, however, applies to a sum of ratios problem different from problem (P).

3. THE GLOBAL OPTIMIZATION ALGORITHM

To solve problem (P), the global optimization algorithm uses a branch and bound search. This search concentrates primarily on the equivalent problem (Q) in order to globally solve problem (P). To present the algorithm, we first need to explain the key operations of this branch and bound procedure.

The branch and bound procedure performs a branching process in outcome space $\Re^{2p}$. This branching process iteratively partitions the rectangle H° containing W into subrectangles. This partitioning process helps the branch and bound procedure identify a location in W that contains a global optimal solution for problem (Q).

To help explain the branching process, we will need the following definition.

DEFINITION 1 [12]. Let V be a subset of $\Re^{2p}$, and let J be a finite set of indices. A set $\{M_j \mid j \in J\}$ of subsets of V is called a *partition* of V when

(a) $V = \cup_{j \in J} M_j$ and

(b) $M_i \cap M_j = \partial_r M_i \cap \partial_r M_j$ for all $i, j \in J, i \neq j$, where $\partial_r M_i$ denotes the (relative) boundary of M_i.

During each iteration of the algorithm, a more refined partition is constructed of a portion of H° that cannot yet be excluded from consideration in the search for a global optimal solution for problem (Q). The initial partition P° consists simply of H°, since at the beginning of the branch and bound procedure, no portion of H° can as yet be excluded from consideration.

At the beginning of a typical Step k of the algorithm, where $k \geq 1$, a partition P^{k-1} is available from the previous step. The partition P^{k-1} consists of $2p$-dimensional rectangles whose union defines a subset of H° that at the end of Step $k-1$ cannot yet be excluded from the branch and bound search. Also available is a rectangle H^{k-1} of P^{k-1} that was chosen in Step $k-1$ for further examination.

During a typical Step k, the branching process subdivides H^{k-1} into two $2p$-dimensional subrectangles of equal volume. This subdivision is accomplished by a process called rectangular bisection. To explain this process, suppose that H is a $2p$-dimensional rectangle in R^{2p} given by

$$H = \{(y, z) \in \Re^{2p} \mid h_y^1 \leq y \leq h_y^2, h_z^1 \leq z \leq h_z^2\}, \tag{4}$$

where $h_y^1, h_y^2, h_z^1, h_z^2 \in \Re^p$, $h_y^1 < h_y^2$ and $h_z^1 < h_z^2$. Suppose that

$$\max\{(h_y^2)_i - (h_y^1)_i, (h_z^2)_i - (h_z^1)_i \mid i \in I\}$$

is achieved by $(h_v^2)_k - (h_v^1)_k$, where $k \in I$ and v equals either y or z. Let $(m_v)_k \in \Re$ be defined by

$$(m_v)_k = [(h_v^1)_k + (h_v^2)_k]/2.$$

Then $\{H^1, H^2\}$ is called a *rectangular bisection* of H, where H^1 and H^2 are the $2p$-dimensional rectangles given by

$$\begin{aligned} H^1 = \{(y, z) \in \Re^{2p} \mid {} & (h_y^1)_i \leq y_i \leq (h_y^2)_i, \text{ if } v \neq y \text{ or } i \neq k, \\ & (h_z^1)_i \leq z_i \leq (h_z^2)_i, \text{ if } v \neq z \text{ or } i \neq k, \\ & (h_v^1)_k \leq v_k \leq (m_v)_k\} \end{aligned}$$

and

$$\begin{aligned} H^2 = \{(y, z) \in \Re^{2p} \mid {} & (h_y^1)_i \leq y_i \leq (h_y^2)_i, \text{ if } v \neq y \text{ or } i \neq k, \\ & (h_z^1)_i \leq z_i \leq (h_z^2)_i, \text{ if } v \neq z \text{ or } i \neq k, \\ & (m_v)_k \leq v_k \leq (h_v^2)_k\}. \end{aligned}$$

Notice that the rectangular bisection $\{H^1, H^2\}$ is formed by subdividing the longest edge of H at the midpoint. From Horst and Tuy [12], the rectangular bisection $\{H^1, H^2\}$ of H forms as partition H in the sense of Definition 1.

Let $\{\overline{H}, \overline{\overline{H}}\}$ denote the rectangular bisection of H^{k-1} formed by the branching process in Step k. Then the partition P^k of the portion $H^\circ \backslash F$ of H° not yet excluded from consideration is

$$P^k = \{H \in \{(P^{k-1} \backslash \{H^{k-1}\}) \cup \{\overline{H}, \overline{\overline{H}}\}\} \mid H \notin F\},$$

where F denotes the current set of rectangles that have been eliminated from consideration by the branch and bound search. We will see how F is derived and updated later.

For purposes of demonstrating convergence, we will need to use a result concerning the rectangular bisection process that, for convenience, we state here. This result is from Horst and Tuy [12].

THEOREM 3. *Let $\{H^i\}$ denote a sequence of $2p$-rectangles such that for each i, $H^{i+1} \subset H^i$ and H^{i+1} is generated from H^i by the rectangular bisection process. Then for some single point q, $\cap_i H^i = \lim_i H^i = \{q\}$.*

In the terminology of global optimization, Theorem 3 states that the rectangular bisection process is an *exhaustive* subdivision process [12].

The branch and bound procedure computes three types of bounds. The first is a local lower bound $\mathrm{LB}(H)$ for the objective function g of problem (Q) over $(W \cap H)$, where H is a given $2p$-dimensional rectangle generated by the branching process. In the algorithm, $\mathrm{LB}(H)$ is computed for $H = H^\circ$ and for each rectangle H formed via rectangular bisection of H^{k-1} for each $k = 1, 2, \ldots$.

In Step k, $k \geq 1$ of the procedure, H^{k-1} is subdivided into two rectangles via rectangular bisection. We may assume that each of these rectangles is of the form of H given in (4). Then, for $k \geq 0$, the local lower bound $\mathrm{LB}(H)$ for g over $(W \cap H)$ that is computed by the procedure is given by

$$\mathrm{LB}(H) = \max\{\widehat{\mathrm{LB}}(H), \mathrm{LB}(H^{k-1})\}, \tag{5}$$

where, for $H = H^\circ$, we set $\mathrm{LB}(H^{-1}) = -\infty$, and the rule for computing $\widehat{\mathrm{LB}}(H)$ depends, in general, upon the solution of two optimization problems.

The first optimization problem (DH) that must be solved to compute $\widehat{\mathrm{LB}}(H)$ is given by

$$D_H = \max\left[\prod_{i=1}^{p} z_i\right]^{1/p}, \qquad \text{s.t. } (y, z) \in W \cap H. \tag{DH}$$

Notice in problem (DH) that $(W \cap H)$ is a convex set, and $(W \cap H) \subseteq \{(y, z) \in \Re^{2p} \mid y \geq 0, z \geq 0\} \doteq \Re_+^{2p}$. From [2, 5], $t(z) \doteq [\prod_{i=1}^{p} z_i]^{1/p}$ is a concave function on the interior of $\Re_+^{2p}$. From convexity theory, since $t(z)$ is continuous on $\Re_+^{2p}$, it follows that $t(z)$ is concave on $\Re_+^{2p}$ as well. As a result, problem (DH) can be solved by any of a number of well-known convex programming methods [3]. The optimal objective function value D_H of problem (DH) is an upper bound for the value of the denominator of g over $(W \cap H)$. If problem (DH) is infeasible, we set $\widehat{\mathrm{LB}}(H) = +\infty$. In this case, no second optimization problem is needed.

When problem (DH) is feasible, a second optimization problem must be solved to compute $\widehat{\mathrm{LB}}(H)$. To understand this problem, we need to present a result given in [4]. Towards this end, suppose that $m \geq 2$, suppose that M is a rectangle given by

$$M = \{w \in \Re^m \mid a \leq w \leq b\},$$

where $a, b \in \Re^m$ and $0 \leq a < b$, and let the function gm: $\Re^m \to \Re$ be defined for each $w \in M$ by

$$gm(w) = \prod_{i=1}^{m} w_i.$$

Then the result is as follows.

THEOREM 4. *Let q: $\Re^m \to \Re$ be defined for each $w \in \Re^m$ by*

$$q(w) = \max\{q_1(w), q_2(w)\},$$

where $q_1(w)$ and $q_2(w)$ are affine functions of w given by the formulas

$$q_1(w) = \sum_{j=1}^{m} \left[\prod_{\substack{i=1 \\ i \neq j}}^{m} a_i \right] w_j - (m-1) \left[\prod_{i=1}^{m} a_i \right]$$

and

$$q_2(w) = \sum_{j=1}^{m} \left[\prod_{\substack{i=1 \\ i \neq j}}^{m} b_i \right] w_j - (m-1) \left[\prod_{i=1}^{m} b_i \right],$$

respectively. Then $q(w) \leq gm(w)$ for all $w \in M$.

Notice that for $m = 2$, the underestimating function q defined in Theorem 4 for gm reduces to the *convex envelope* of gm over M. For details see [4, 12].

The second optimization problem (NH) that must be solved to compute $\widehat{\mathrm{LB}}(H)$ when problem (DH) is feasible is given by

$$N_H = \min \sum_{i=1}^{p} s_i \tag{NH}$$

subject to

$$s_i \geq \left[\prod_{\substack{k=1\\k\neq i}}^{p}(h_z^1)_k\right] y_i + \sum_{\substack{k=1\\k\neq i}}^{p}\left[(h_y^1)_i\left(\prod_{\substack{j=1\\j\neq i,\,k}}^{p}(h_z^1)_j\right)\right] z_k - (p-1)(h_y^1)_i$$

$$\times\left(\prod_{\substack{k=1\\k\neq i}}^{p}(h_z^1)_k\right), \qquad i = 1, 2, \ldots, p, \tag{6}$$

$$s_i \geq \left[\prod_{\substack{k=1\\k\neq i}}^{p}(h_z^2)_k\right] y_i + \sum_{\substack{k=1\\k\neq i}}^{p}\left[(h_y^2)_i\left(\prod_{\substack{j=1\\j\neq i,\,k}}^{p}(h_z^2)_j\right)\right] z_k - (p-1)(h_y^2)_i$$

$$\times\left(\prod_{\substack{k=1\\k\neq i}}^{p}(h_z^2)_k\right), \qquad i = 1, 2, \ldots, p, \tag{7}$$

$$(y, z) \in W \cap H.$$

From the definition of g and Theorem 4, we see that the optimal value N_H of problem (NH) underestimates the pth power of the numerator of g over $(W \cap H)$. Since $(W \cap H)$ is a convex set and inequality constraints (6) and (7) are linear, problem (NH), like problem (DH), can be solved by any of a number of convex programming methods [3].

The lower bound $\widehat{\mathrm{LB}}(H)$ in (5) for g over $(W \cap H)$ is given by

$$\widehat{\mathrm{LB}}(H) = (N_H)^{1/p}/D_H,$$

where N_H and D_H are the optimal values of the convex programming problems (NH) and (DH), respectively. The local lower bound $\mathrm{LB}(H)$ computed in the bounding procedure is then given by (5).

Notice for $k \geq 0$ in (5) that if $\widehat{\mathrm{LB}}(H) < \mathrm{LB}(H^{k-1})$, then the local lower bound $\mathrm{LB}(H)$ is set equal to $\mathrm{LB}(H^{k-1})$, rather than to $\widehat{\mathrm{LB}}(H)$. It is easy to show that LB (H^{k-1}) is also a valid lower bound for g over $(W \cap H)$. One purpose of (5) is to use a local lower bound $\mathrm{LB}(H)$ for g over $(W \cap H)$ that is as large as possible. Another is to ensure that $\mathrm{LB}(H) \geq \mathrm{LB}(H^{k-1})$, which is important for the convergence of the algorithm.

The second bound computed by the branch and bound procedure is a global lower bound LB for the optimal objective function value $(vm)^{1/p}$ of problem (Q). In Step 0 this lower bound is set equal to the local lower bound $\mathrm{LB}(H^\circ)$. Since $\mathrm{LB}(H^\circ)$ is a global lower bound for g over $(W \cap H^\circ)$ and H° contains W, $\mathrm{LB}(H^\circ)$ is a global lower bound for g over

W; i.e., $\mathrm{LB}(H^\circ) \leq (vm)^{1/p}$. For each $k \geq 1$, in Step k, this lower bound LB is computed via the equation

$$\mathrm{LB} = \min\{\mathrm{LB}(\widetilde{H}) \mid \widetilde{H} \in P^k\}. \tag{8}$$

Subsequently, a rectangle $H^k \in P^k$ such that $\mathrm{LB} = \mathrm{LB}(H^k)$ is identified. Unless the procedure terminates, this is the rectangle that will be bisected in the next step of the search.

The third bound computed by the branch and bound procedure is a global upper bound for the optimal objective function value $(vm)^{1/p}$ of problem (Q). For each $k \geq 0$, this global upper bound UB_k for $(vm)^{1/p}$ is given by

$$\mathrm{UB}_k = [f(x^c)]^{1/p}, \tag{9}$$

where $x^c \in X$ is the *incumbent* feasible solution for problem (P); i.e., among all feasible solutions for problem (P) found through any point in the procedure, $x = x^c$ achieves the smallest value of $[f(x)]^{1/p}$. Feasible solutions for problem (P) are found as the convex programming problems (DH) and (NH) are solved.

The set F in the branch and bound procedure is the set of *fathomed* rectangles H. When, in some Step $k \geq 1$, the algorithm detects that

$$\mathrm{LB}(H) \geq \mathrm{UB}_k \tag{10}$$

for some rectangle H of the form (4) created by the branching process, the rectangle H is fathomed (eliminated from consideration); i.e., it is added to F. Rectangles H in F need not be subdivided nor searched further for a global optimal solution for problem (Q). This is because from (10), for any such rectangle H, if $(y, z) \in (W \cap H)$, then

$$g(y, z) \geq \mathrm{LB}(H) \geq \mathrm{UB}_k = [f(x^c)]^{1/p},$$

where the first inequality follows from the validity of the local lower bound $\mathrm{LB}(H)$ for g over $(W \cap H)$, and the equality is from (9). Thus, rectangles in F cannot contain solutions superior to the solution $(y^c, z^c) \doteq [n(x^c), d(x^c)]$ for problem (Q).

The validity of LB in (8) as a global lower bound for the optimal value $(vm)^{1/p}$ of problem (Q) can be shown quite easily by using Theorem 4 and applying standard branch and bound arguments from global optimization [12]

Based upon the results and key operations given in this section, the global optimization algorithm for problem (P) may be stated as follows.

GLOBAL OPTIMIZATION ALGORITHM.

Step 0. Choose $\epsilon \geq 0$. Find a rectangle $H^\circ \subseteq \Re^{2p}$ as specified by (1)–(3). Set $P^\circ = \{H^\circ\}$, $F = \emptyset$, and $\mathrm{LB} = \mathrm{LB}(H^\circ)$, where $\mathrm{LB}(H^\circ)$ is found via (5) with H set equal to H°. Set $UB_\circ = \min\{[f(x)]^{1/p} \mid x \in G^\circ\}$, where G° denotes the set of feasible solutions for problem (P) found in the course of computing $\mathrm{LB}(H^\circ)$. Set x^c equal to any of the solutions $x \in G^\circ$ such that $[f(x)]^{1/p} = \mathrm{UB}_\circ$. Set $k = 1$ and go to Step k.

Step $k(k \geq 1)$.

Step k.1. Set $\mathrm{UB}_k = \mathrm{UB}_{k-1}$. If $\mathrm{UB}_k^p - \mathrm{LB}^p \leq \epsilon$, stop: x^c is a global ϵ-optimal solution for problem (P). Otherwise continue.

Step k.2. Subdivide the rectangle $H^{k-1} \subseteq \Re^{2p}$ into two $2p$-dimensional rectangles $\overline{H}, \overline{\overline{H}} \subseteq \Re^{2p}$ via the rectangular bisection process.

Step k.3. For each of $H = \overline{H}$ and $H = \overline{\overline{H}}$, compute $\mathrm{LB}(H)$ via (5). For each of $H = \overline{H}$ and $H = \overline{\overline{H}}$, if $\mathrm{LB}(H) \geq \mathrm{UB}_k$, set $F = F \cup \{H\}$. If $\overline{H}, \overline{\overline{H}} \in F$, set $G^k = \emptyset$; otherwise let G^k denote the set of feasible solutions for problem (P) found in the course of computing $\mathrm{LB}(\overline{H})$ and $\mathrm{LB}(\overline{\overline{H}})$.

Step k.4. Set $\mathrm{UB}_k = \min\{\mathrm{UB}_k \cup \{[f(x)]^{1/p} \mid x \in G^k\}\}$, and set x^c equal to any solution x such that $[f(x)]^{1/p} = \mathrm{UB}_k$.

Step k.5. Set $P^k = \{H \in [(P^{k-1} \backslash \{H^{k-1}\}) \cup \{\overline{H}, \overline{\overline{H}}\}] \mid H \notin F\}$.

Step k.6. Set $\mathrm{LB} = \min\{\mathrm{LB}(H) \mid H \in P^k\}$. Choose any $\widehat{H} \in P^k$ such that $\mathrm{LB} = \mathrm{LB}(\widehat{H})$. Set $H^k = \widehat{H}$, set $k = k + 1$, and go to Step k.

Notice that in Step 0, the set G° will contain two feasible solutions for problem (P), one from solving problem (DH) with $H = H^\circ$, and one from solving problem (NH) with $H = H^\circ$. For $k \geq 1$, the set G^k computed in Step k.3 may contain up to four feasible solutions for problem (P).

Notice also that the algorithm concentrates primarily on globally solving problem (Q). However, in Step 0 and, for each $k \geq 1$, in Step k.4, the algorithm finds incumbent vectors x^c that are feasible solutions for problem (P). It is through these incumbent solutions x^c that the algorithm globally solves problem (P), as we shall see in the next section.

4. CONVERGENCE AND INITIALIZATION

The main convergence property of the global optimization algorithm is given in the following result.

THEOREM 5. *Suppose that the global optimization algorithm is infinite. For each* $k \geq 0$, *let* x^k *denote the incumbent solution vector value* x^c *that exists at the end of Step* k.4 *of the algorithm. Then* $\{x^k\}$ *is a sequence of feasible solutions for problem* (P), *and every accumulation point of* $\{x^k\}$ *is a global optimal solution for problem* (P). *Furthermore*

$$\lim_{k\to\infty} (\mathrm{UB}_k)^p = \lim_{k\to\infty} [\mathrm{LB}(H^k)]^p = vm.$$

Proof. For each $j \geq 0$, let $(x^{D,j}, y^{D,j}, z^{D,j})$ and $(x^{N,j}, y^{N,j}, z^{N,j}, s^{N,j})$ denote the optimal solutions found to problems (DH) and (NH), respectively, for $H = H^j$. Then, for each j, since $H^j \subseteq H^\circ$, $(y^{D,j}, z^{D,j})$ and $(y^{N,j}, z^{N,j})$ are feasible solutions for problem (Q) for which $x^{D,j}, x^{N,j} \in X$, $y^{D,j} \geq n(x^{D,j})$, $z^{D,j} \leq d(x^{D,j})$, $y^{N,j} \geq n(x^{N,j})$, and $z^{N,j} \leq d(x^{N,j})$. Since the algorithm is infinite, we may assume without loss of generality that $H^{j+1} \subset H^j$ for all $j \geq 0$. By Theorem 3, this implies that $\cap_j H^j = \lim_{j\to\infty} H^j = \{(\bar{y}, \bar{z})\}$ for some $(\bar{y}, \bar{z}) \in \Re^{2p}$. Therefore,

$$\lim_j (y^{D,j}, z^{D,j}) = \lim_j (y^{N,j}, z^{N,j}) = (\bar{y}, \bar{z}). \tag{11}$$

Let $\bar{x}$ be an accumulation point of $\{x^k\}$, and assume for notational convenience and without loss of generality that $\lim_k x^k = \bar{x}$. For each k, x^k equals either $x^{D,k}$ or $x^{N,k}$. This implies that $x^k \in X$ for each k and, from the closedness of X and the continuity of $n_i(x), d_i(x)$, $i \in I$, that $(\bar{y}, \bar{z}) \in W$, $\bar{x} \in X$, $\bar{y} \geq n(\bar{x})$ and $\bar{z} \leq d(\bar{x})$. Therefore, $(\bar{y}, \bar{z})$ is a feasible solution for problem (Q) and $\bar{x}$ is a feasible solution for problem (P).

From Section 3, since $H^{k+1} \subset H^k$ for all k, the sequence $\{\mathrm{LB}(H^k)\}$ is nondecreasing and bounded above by $(vm)^{1/p}$. For each k, from (5), $\mathrm{LB}(H^k) \geq \widehat{\mathrm{LB}}(H^k)$. From Section 3, for each k,

$$\widehat{\mathrm{LB}}(H^k) = (N_{H^k})^{1/p}/D_{H^k}.$$

Taken together, the previous three observations imply that for each k,

$$\begin{aligned} v_m^{1/p} &\geq \mathrm{LB}(H^k) & (12)\\ &\geq \widehat{\mathrm{LB}}(H^k) \\ &= (N_{H^k})^{1/p}/D_{H^k} \\ &= \left(\sum_{i=1}^{p} s_i^{N,k}\right)^{1/p} \Big/ \left(\prod_{i=1}^{p} z_i^{D,k}\right)^{1/p}, & (13)\end{aligned}$$

where the last equation follows from the definitions of problems (DH) and (NH) for $H = H^k$. For each k, let H^k be given by

$$H^k = \{(y, z) \in \Re^{2p} \mid (h_y^1)^k \leq y \leq (h_y^2)^k, (h_z^1)^k \leq z \leq (h_z^2)^k\}.$$

From problem (NH) with $H = H^k$, we know that for each k and each $i \in I$,

$$s_i^{N,k} = \max\left[F_1\left(y_i^{N,k}, z^{i,N,k}\right), F_2\left(y_i^{N,k}, z^{i,N,k}\right)\right], \tag{14}$$

where $z^{i,N,k} = (z_1^{N,k}, z_2^{N,k}, \ldots, z_{i-1}^{N,k}, z_{i+1}^{N,k}, \ldots, z_p^{N,k})$, and F_1 and F_2 are the linear functions given in the right-hand sides of the inequalities (6) and (7).

For each k, from (12)–(14), it follows that

$$vm^{1/p} \geq \mathrm{LB}(H^k) \geq \left[\sum_{i=1}^{p} \max \left(F_1\left(y_i^{N,k}, z^{i,N,k}\right), \right.\right.$$
$$\left.\left. F_2\left(y_i^{N,k}, z^{i,N,k}\right)\right) \Big/ \prod_{i=1}^{p} z_i^{D,k} \right]^{1/p}.$$

By taking limits over k in these two inequalities, from (6) and (7), the fact that $\{\mathrm{LB}(H^k)\}$ is nondecreasing, and the fact that $\lim_k H^k = \{(\bar{y}, \bar{z})\}$, we obtain that

$$(vm)^{1/p} \geq \lim_k \mathrm{LB}(H^k) \geq \left[\sum_{i=1}^{p} \max \left(\bar{y}_i \prod_{\substack{j=1 \\ j\neq i}}^{p} \bar{z}_j, \bar{y}_i \prod_{\substack{j=1 \\ j\neq i}}^{p} \bar{z}_j \right) \Big/ \prod_{j=1}^{p} \bar{z}_j \right]^{1/p}$$
$$= g(\bar{y}, \bar{z}).$$

Since $(\bar{y}, \bar{z})$ is a feasible solution for problem (Q), by Theorem 2, $g(\bar{y}, \bar{z}) \geq vm^{1/p}$. Together with the previous sentence, this implies that

$$vm^{1/p} = \lim_k \mathrm{LB}(H^k) = g(\bar{y}, \bar{z}), \tag{15}$$

so that by Theorem 2, $(\bar{y}, \bar{z})$ is a global optimal solution for problem (Q). By Theorem 2, since $\bar{x} \in X, \bar{y} \geq n(\bar{x})$ and $\bar{z} \leq d(\bar{x}), \bar{x}$ is a global optimal solution for problem (P).

From (9) and the definition of $x^k, \mathrm{UB}_k = [f(x^k)]^{1/p}$ for each k. Since f is a continuous function and $\bar{x}$ is a global optimal solution for problem (P), this implies that

$$\lim_k \mathrm{UB}_k = [f(\bar{x})]^{1/p} = vm^{1/p}.$$

By (15), since $\bar{x}$ is a global optimal solution for problem (P), this completes the proof.

A solution $\bar{x}$ is called a *global ϵ-optimal* solution for problem (P), where $\epsilon \geq 0$ is a given number, when $\bar{x} \in X$ and $vm \leq f(\bar{x}) \leq vm + \epsilon$. By using Theorem 5, the following result can be easily shown.

COROLLARY 1. *If $\epsilon > 0$, the algorithm is finite. In this case, upon termination, x^c is a global ϵ-optimal solution for problem* (P).

In Section 3 we mentioned that to help implement the algorithm, each of the occurrences of problems (DH) and (NH) may be solved by any of a number of convex programming methods. The other main implementation issue concerns finding the initial rectangle H° in Step 0 as specified by (1)–(3).

For each $i \in I$, finding a value for N_i that satisfies (1) calls in general for finding an overestimate of the optimal value of a convex maximization problem. An efficient procedure for accomplishing this that is based upon convex programming is given in [4]. Of course, for each $i \in I$, if $n_i(x)$ is linear, then the optimal value in (1) can be found by any of a number of convex programming methods [3]. For each $i \in I$, since $d_i(x)$ is a finite, concave function, the value of M_i in (2) can be found by convex programming.

5. CONCLUDING REMARKS

We have presented and shown the convergence properties of an algorithm for finding a global optimal solution to the nonlinear sum of ratios problem (P). This global optimization problem is particularly challenging to solve. Even in the special case where $p = 2$, X is a polyhedron, the three functions $n_1(x)$, $n_2(x)$, and $d_2(x)$ are linear functions, and $d_1(x)$ is a constant, problem (P) generally possesses many local optima that are not global optima.

The global optimization algorithm for problem (P) that we have presented uses a branch and bound search procedure. This procedure solves problem (P) by concentrating primarily upon solving the equivalent problem (Q). Since problem (Q) is defined in the outcome space $\Re^{2p}$ of problem (P), and in many applications $2p$ is smaller than n, it is expected that this approach will economize on computations. In addition, the branch and bound search procedure is structured so that it can be implemented by solving only convex programming problems. This is another potentially useful characteristic of the algorithm, since any of a number of standard methods can be used to solve these problems.

It is hoped that in practice, the proposed algorithm and the ideas used in it will offer valuable tools for solving nonlinear sum of ratios problems.

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